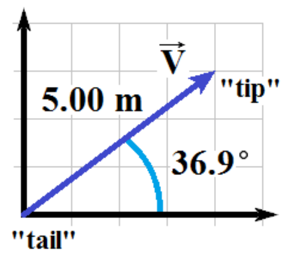
Part 3: Vectors

College Physics (Openstax): Chapters 2
Physics (Giancoli): Chapters 3

Vectors

- A **vector quantity** is characterized by two properties (magnitude, directions = 2 numbers).
- A **scalar quantity** is characterized by a single property (magnitude = 1 number).

Vectors can have more than two properties/numbers, but (for the moment) we are restricting ourselves to 2-dimensional vectors, which are limited to 2.



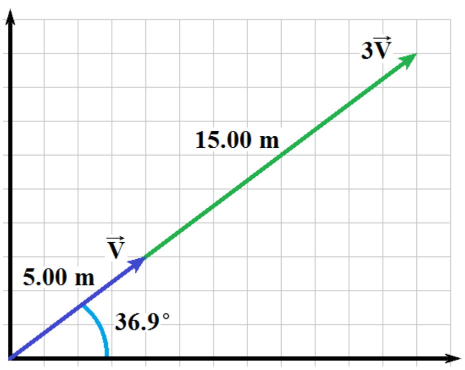
Example Vector:

$\vec{V} = V\angle\theta_V = (5.00\text{ m}) \angle 36.9°$

Magnitude (length): $|\vec{V}| = V = (5.00\text{ m})$

Angle: $\theta_V = 36.9°$

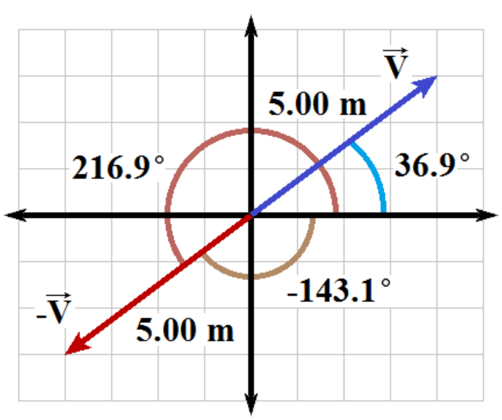
Vector Math



Multiply a Vector by a Positive Scalar

- Multiply the magnitude (V) by the scalar (a)
- $(3.00)\vec{V} = (3.00 \times V)\angle\theta_V =$ $(3.00)(5.00\text{ m}) \angle 36.9° = 15.0\text{ m} \angle 36.9°$

This changes the length without any change to the direction.



Multiply a Vector by "–1"

- Add (or subtract) 180° to (or from) the angle.
- $-1\,\vec{V} = -V\angle\theta_V = V\angle\theta_V \pm 180° =$ $5.00\text{ m} \angle 216.9°$ or $5.00\text{ m} \angle -143.1°$

Adding 180° and subtracting 180° effectively do the same thing (answers will differ by 360°)

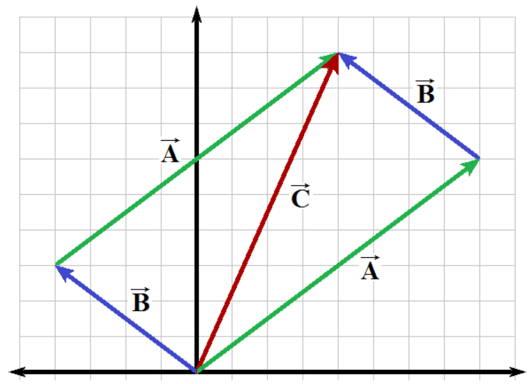
If you turn around, it doesn't matter if you turned to the right or the left. The result is the same.

While these are mathematically equivalent, most homework services will only accept one (they tell you the range of values, typically 0° to 360° or -180° to 180°, that are acceptable).

Multiply a Vector by a Negative Scalar $\Rightarrow$ break it into a "-1" and a positive scalar

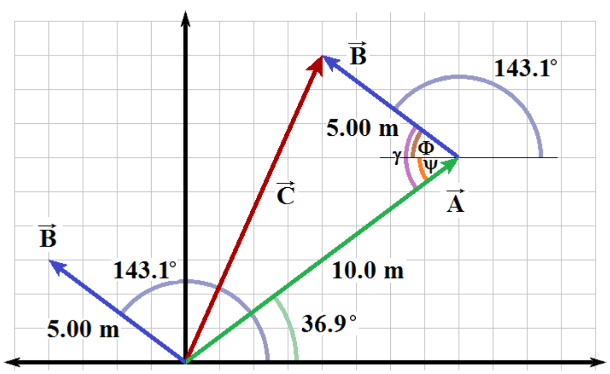
- $(-3.00)\vec{V} = (3.00 \times V)\angle\theta_V \pm 180° = 15.0\text{ m} \angle 216.9°\text{ or } -143.1°$

Adding Vectors



- Leave your first vector where it is.
- Slide the second vector up so its "tail" starts at the "tip" of the first vector.
- The sum of these two vectors is a vector that starts at the tail of the first and goes to the tip of the second.
- $\vec{A} + \vec{B} = \vec{C}$
- The sum of two vectors is called the resultant (often denoted by $\vec{R}$ rather than $\vec{C}$)
- Vector addition is commutative. $\vec{A} + \vec{B} = \vec{B} + \vec{A} = \vec{C}$
- One method of determining the magnitude and angle of the resultant is to use geometry, the law of cosines, and/or the law of sines.

Example: If $\vec{A} = 10.0\angle\ 36.9°$ and $\vec{B} = 5.00\angle 143.1°$, determine the magnitude and angle of $\vec{C} = \vec{A} + \vec{B}$.



- First draw vectors $\vec{A}$ and $\vec{B}$.
- Draw a second vector $\vec{B}$ starting at the tip of vector $\vec{A}$.
- Draw in vector $\vec{C}$ from the tail of $\vec{A}$ to the tip of $\vec{B}$.

If you know the length of two sides of a triangle and the angle in between them, the law of cosines allows you to determine the length of the third side.

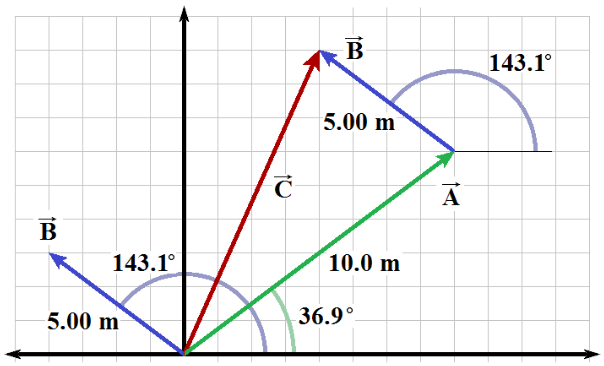
- We need to find angle γ, which is the sum of angles Φ and Ψ.
  - $\Phi = 180° - 143.1 = 36.9°$
  - $\Psi = \theta_A = 36.9°$
  - $\gamma = \Psi + \Phi = 36.9° + 36.9° = 73.8°$

- Law of Cosines: $C^2 = A^2 + B^2 - 2AB\text{Cos}(\gamma)$
  - $C^2 = (10.0\text{ m})^2 + (5.00\text{ m})^2 - 2(10.0\text{ m})(5.00\text{ m})\text{Cos}(73.8°)$
  - $C^2 = 100.\text{ m}^2 + 25.0\text{ m}^2 - (100.\text{ m}^2)\text{Cos}(73.8°)$
  - $C^2 = 100\text{ m}^2 + 25.0\text{ m}^2 - 27.9\text{ m}^2 = 97.1\text{ m}^2$
  - $C = \sqrt{97.1\text{ m}^2} = 9.85398\text{ m} = 9.85\text{ m}$

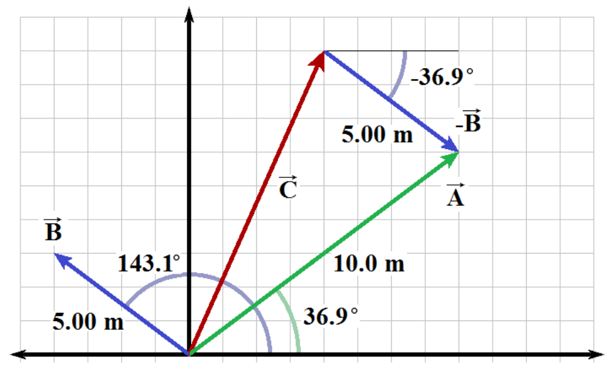
- Law of Sines: $\frac{\text{Sin}\,\alpha}{A} = \frac{\text{Sin}\,\beta}{B} = \frac{\text{Sin}\,\gamma}{C}$
  - $\text{Sin}\,\beta = \frac{B}{C}\text{Sin}\,\gamma$
  - $\beta = \text{Sin}^{-1}\left(\frac{B}{C}\text{Sin}\,\gamma\right) = \text{Sin}^{-1}\left\{\frac{5.00\text{ m}}{9.85398\text{ m}}\text{Sin}(73.8°)\right\} = 29.1608°$
  - $\theta_C = \beta + \theta_A = 29.1608° + 36.9° = 66.1°$
- $\vec{C} = 9.85\text{ m} \angle 66.1°$

Subtracting Vectors

- To subtract, multiply a vector by "-1" and add: $\vec{C} - \vec{B} = \vec{C} + (-\vec{B})$
- In the previous example, since $\vec{A} + \vec{B} = \vec{C}$, then $\vec{A} = \vec{C} - \vec{B}$



Addition $\vec{A} + \vec{B} = \vec{C}$



Subtraction $\vec{A} = \vec{C} - \vec{B} = \vec{C} + (-\vec{B})$

Example: Vector $\vec{A}$ has a magnitude of 12.3 km and points due west. Vector $\vec{B}$ points due north. If $\vec{A} + \vec{B}$ has a magnitude of 15.0 km, then a) what is the magnitude of $\vec{B}$? And b) what is the direction of $\vec{A} + \vec{B}$ relative to due west? If $\vec{A} - \vec{B}$ has a magnitude of 15.0 km, then c) what is the magnitude of $\vec{B}$? And d) what is the direction of $\vec{A} - \vec{B}$ relative to due west?

a) $\vec{C} = \vec{A} + \vec{B}$ $\quad A^2 + B^2 = C^2$ $\quad B^2 = C^2 - A^2$

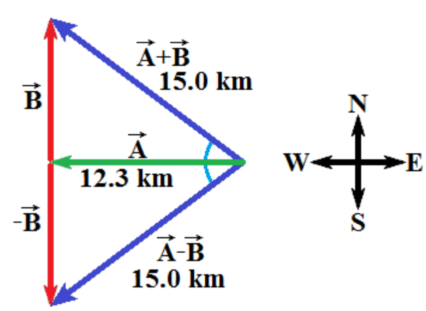
$B = \sqrt{C^2 - A^2} = \sqrt{(15.0\text{ km})^2 - (12.3\text{ km})^2} = 8.6\text{ km}$

b) $\theta = \text{Cos}^{-1}\left(\frac{A}{C}\right) = \text{Cos}^{-1}\left(\frac{12.3\text{ km}}{15.0\text{ km}}\right) = 34.9°$ N of W

c) $\vec{C} = \vec{A} - \vec{B}$ $\quad A^2 + B^2 = C^2$ $\quad B^2 = C^2 - A^2$

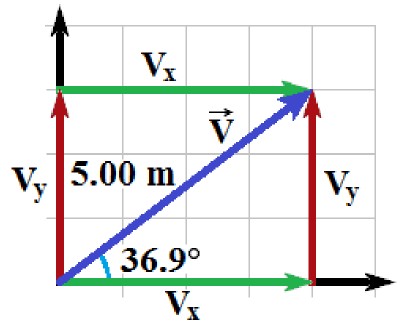
$B = \sqrt{C^2 - A^2} = \sqrt{(15.0\text{ km})^2 - (12.3\text{ km})^2} = 8.6\text{ km}$

d) $\theta = \text{Cos}^{-1}\left(\frac{A}{C}\right) = \text{Cos}^{-1}\left(\frac{12.3\text{ km}}{15.0\text{ km}}\right) = 34.9°$ S of W



Vector Components

- Vectors can be broken into components which point along the axis of a reference frame (coordinate system).
- Collectively, the components are equivalent to the original vector.
- In some cases, computations are more easily done with components than the original vector.

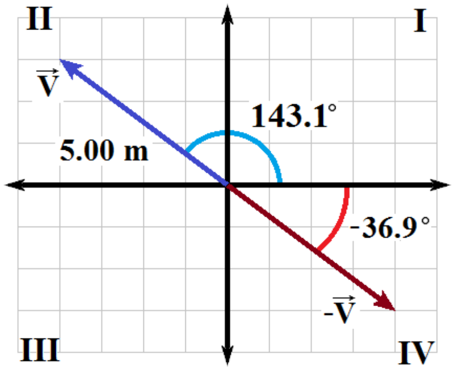


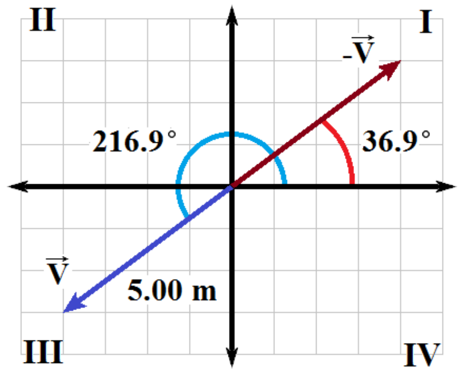
- To determine the horizontal and vertical components of a vector (V_x and V_y) use trigonometry.
  - $\mathrm{Cos}(\theta_V) = \frac{\text{Adjacent Side}}{\text{Hypotenuse}} = \frac{V_x}{V}$ $\quad V_x = V \cdot \mathrm{Cos}(\theta_V)$
  - $\mathrm{Sin}(\theta_V) = \frac{\text{Opposite Side}}{\text{Hypotenuse}} = \frac{V_y}{V}$ $\quad V_y = V \cdot \mathrm{Sin}(\theta_V)$

The previous equations are only valid when the angle (θ_V) is given with respect to the positive x-axis. In many cases angles are defined differently.

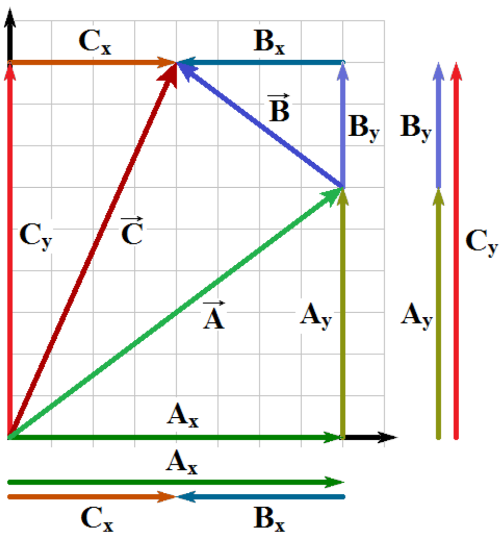
- For example, if $\vec{V}$ = (5.00 m)∠36.9°
  - $V_x = V \cdot \mathrm{Cos}(\theta_V) = (5.00\ \mathrm{m})\mathrm{Cos}(36.9°) = 4.00\ \mathrm{m}$
  - $V_y = V \cdot \mathrm{Sin}(\theta_V) = (5.00\ \mathrm{m})\mathrm{Sin}(36.9°) = 3.00\ \mathrm{m}$

- To determine the magnitude and angle from the horizontal and vertical components of a vector (V_x and V_y) use the Pythagorean Theorem and an inverse trigonometric function.
  - $V = |\vec{V}| = \sqrt{V_x^2 + V_y^2}$
  - If $V_x > 0$, then $\theta_V = \mathrm{Tan}^{-1}\left(\frac{V_y}{V_x}\right)$
  - If $V_x < 0$, then $\theta_V = \mathrm{Tan}^{-1}\left(\frac{V_y}{V_x}\right) \pm 180°$
  - Calculators determine the quadrant based on the sign of the argument in Tan^{-1}, placing it in quadrant I (angles from 0 to 90°) if V_y/V_x is positive and in quadrant IV (angles from 0 to -90°) if V_y/V_x is negative.
  - This works in quadrant I (where V_x and V_y are positive and the argument, V_y/V_x is positive) and quadrant IV (where V_x is positive, V_y is negative, and the argument, V_y/V_x is negative).
  - In quadrant II (where V_x is negative, V_y is positive, and the argument, V_y/V_x is negative), it returns an answer in quadrant IV, which is the angle of -$\vec{V}$. Adding or subtracting 180° corrects this.
  - In quadrant III (where V_x and V_y are negative, V_y/V_x is positive), it returns an answer in quadrant I, which is the angle of -$\vec{V}$. Adding or subtracting 180° corrects this.





Vector Addition with Components



- Break vectors into components.
- Add corresponding components to get the resultant's components
  - $\vec{A} + \vec{B} = \vec{C}$
  - $A_x + B_x = C_x$
  - $A_y + B_y = C_y$
- Find resultant's magnitude and angle from the resultant's components.

Sometimes it's easier to add components. Sometimes it's easier to use the law of cosines. Learn both.

Example: Find $\vec{C} = \vec{A} + \vec{B}$, when $\vec{A} = (7.28\text{ m})\angle 15.95°$ and $\vec{B} = (5.00\text{ m})\angle 36.87°$.

$$C_x = A_x + B_x = A \cdot \mathrm{Cos}(\theta_A) + B \cdot \mathrm{Cos}(\theta_B)$$

$$C_x = (7.28\text{ m}) \cdot \mathrm{Cos}(15.95°) + (5.00\text{ m}) \cdot \mathrm{Cos}(36.87°) = 7.00\text{ m} + 4.00\text{ m} = 11.00\text{ m}$$

$$C_y = A_y + B_y = A \cdot \mathrm{Sin}(\theta_A) + B \cdot \mathrm{Sin}(\theta_B)$$

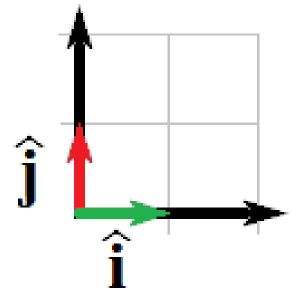
$$C_y = (7.28\text{ m}) \cdot \mathrm{Sin}(15.95°) + (5.00\text{ m}) \cdot \mathrm{Sin}(36.87°) = 2.00\text{ m} + 3.00\text{ m} = 5.00\text{ m}$$

$$C = \sqrt{C_x^2 + C_y^2} = \sqrt{(11.00\,m)^2 + (5.00\,m)^2} = 12.08\,m$$

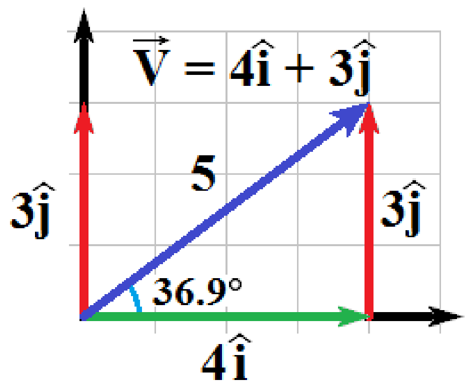
$$\theta_C = \mathrm{Tan}^{-1}\left(\frac{C_y}{C_x}\right) = \mathrm{Tan}^{-1}\left(\frac{5.00\text{ m}}{11.00\text{ m}}\right) = 24.44°$$

$$\vec{C} = 12.08\text{ m} \angle 24.4°$$

Unit Vectors



- Unit vectors are vectors of magnitude 1. Usually these vectors align with coordinate axes.
- Multiplying a unit vector by a scalar creates a vector with length equal to the scalar pointing along a coordinate axis.
- Any vector may be represented as the sum of it's horizontal and vertical components multiplied by their respective unit vectors. $\vec{V} = V\angle\theta_V = V_x\hat{\imath} + V_y\hat{\jmath}$



For example,

$$\vec{V} = V\angle\theta_V = (5.00\text{ m})\angle 36.9° = (4.00\text{ m})\hat{\imath} + (3.00\text{m})\hat{\jmath}$$

- This creates a convenient notation for vector addition.

$$\vec{A} + \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right) = \left(A_x\hat{\imath} + B_x\hat{\imath}\right) + \left(A_y\hat{\jmath} + B_y\hat{\jmath}\right)$$

$$\vec{A} + \vec{B} = \left(A_x + B_x\right)\hat{\imath} + \left(A_y + B_y\right)\hat{\jmath} = C_x\hat{\imath} + C_y\hat{\jmath} = \vec{C}$$

Example: Find the resultant of 3 displacement vectors by means of the component method. The three vectors are $\vec{A}$ = (5.00 m) ∠160°, $\vec{B}$ = (5.00 m) ∠60°, and $\vec{C}$ = (4.00 m) ∠270°.

$$\vec{A} = (5.00\text{ m})\angle 160° = (5.00\text{ m})\text{Cos}(160°)\hat{\imath} + (5.00\text{ m})\text{Sin}(160°)\hat{\jmath} = (-4.70\text{ m})\,\hat{\imath} + (1.71\text{ m})\,\hat{\jmath}$$

$$\vec{B} = (5.00\text{ m})\angle 60° = (5.00\text{ m})\text{Cos}(60°)\hat{\imath} + (5.00\text{ m})\text{Sin}(60°)\hat{\jmath} = (2.50\text{ m})\hat{\imath} + (4.33\text{ m})\hat{\jmath}$$

$$\vec{C} = (4.00\text{ m})\angle 270° = (4.00\text{ m})\text{Cos}(270°)\hat{\imath} + (4.00\text{ m})\text{Sin}(270°)\hat{\jmath} = (-4.00\text{m})\hat{\jmath}$$

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} = \underbrace{\{(-4.70\text{ m})\hat{\imath} + (1.71\text{ m})\hat{\jmath}\}}_{\vec{A}} + \underbrace{\{(2.50\text{ m})\hat{\imath} + (4.33\text{ m})\hat{\jmath}\}}_{\vec{B}} + \underbrace{\{(-4.00\text{m})\hat{\jmath}\}}_{\vec{C}}$$

$$\vec{R} = \{(-4.70\text{ m}) + (2.50\text{ m})\}\hat{\imath} + \{(1.71\text{ m}) + (4.33\text{ m}) + (-4.00\text{m})\}\hat{\jmath}$$

$$\vec{R} = (-2.20\text{ m})\hat{\imath} + (2.04\text{ m})\hat{\jmath}$$

$$R = \left|\vec{R}\right| = \sqrt{R_x^2 + R_y^2} = \sqrt{(2.20\text{ m})^2 + (2.04\text{ m})^2} = 3.00\text{ m}$$

$$\theta_R = \mathrm{Tan}^{-1}\left(\frac{R_y}{R_x}\right) + 180° = \mathrm{Tan}^{-1}\left(\frac{2.04\text{ m}}{-2.20\text{ m}}\right) + 180° = -42.8° + 180° = 137.2°$$

$$\vec{R} = (-2.20\text{ m})\hat{\imath} + (2.04\text{ m})\hat{\jmath} = (3.00\text{ m})\ \angle 137.2°$$

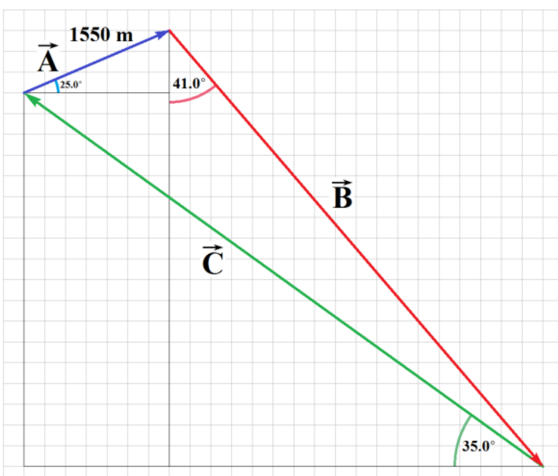
Example: The route followed by a hiker consists of 3 displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Vector $\vec{A}$ is along a measured trail and is 1550m in a direction 25.0° north of east. Vector $\vec{B}$ is not along a measured trail but the hiker uses a compass and knows that the direction is 41.0° east of south. Similarly the direction of vector $\vec{C}$ is 35° north of west. The hiker ends up back where she started, so the resulting displacement is zero (or $\vec{A} + \vec{B} + \vec{C} = 0$). Find the magnitude of vector $\vec{B}$ and vector $\vec{C}$.

We can solve this is two different ways. The first option, by using components, requires one to solve a system of two equations. The second option, using the law of sines, requires a good bit of geometry. We will solve this one both ways.

First make a diagram (not easy when most lengths aren't given).

Solution 1 (by components)

Note: The angles given for $\vec{B}$ and $\vec{C}$ are not defined with respect to the positive x-axis. You must make triangles and use trigonometry to find components.



A_x = (1550 m)Cos(25.0°) = 1404.8 m

A_y = (1550 m)Sin(25.0°) = 655.1 m

B_x = B·Sin(41.0°) *{B_x is the opposite side}*

B_y = -B·Cos(41.0°) *{B_y is pointed down}*

C_x = -C·Cos(35.0°) *{C_x is pointed left}*

C_y = C·Sin(35.0°)

$$A_x + B_x + C_x = 0$$

$$1404.8\text{ m} + B\cdot\mathrm{Sin}(41.0°) - C\cdot\mathrm{Cos}(35.0°) = 0$$

$$A_y + B_y + C_y = 0$$

$$655.1\text{ m} - B\cdot\mathrm{Cos}(41.0°) + C\cdot\mathrm{Sin}(35.0°) = 0$$

2 equations, 2 unknowns (B and C).

Multiply the first equation by Sin (35.0°), multiply the second equation by Cos (35.0°), and then add he two equations. The C terms will disappear.

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°) - C\cdot\text{Cos}(35.0°)\cdot\text{Sin}(35.0°) = 0$$
$$\underline{(655.1\text{ m})\cdot\text{Cos}(35.0°) - B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) + C\cdot\text{Cos}(35.0°)\cdot\text{Sin}(35.0°) = 0}$$
$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) + B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°) - B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) = 0$$

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) = B\cdot\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - B\cdot\text{Sin}(41.0°)\cdot\text{Sin}(35.0°)$$

$$(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°) = B\cdot\{\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - \text{Sin}(41.0°)\cdot\text{Sin}(35.0°)\}$$

$$B = \{(1404.8\text{ m})\cdot\text{Sin}(35.0°) + (655.1\text{ m})\cdot\text{Cos}(35.0°)\}/\{\text{Cos}(41.0°)\cdot\text{Cos}(35.0°) - \text{Sin}(41.0°)\cdot\text{Sin}(35.0°)\}$$

$$B = \{805.76\text{ m} + 536.63\text{ m}\}/\text{Cos}(76.0°) = 1342.39\text{ m}/\text{Cos}(76.0°) = 5548.86\text{ m} \Rightarrow 5550\text{ m}$$

Solve first equation for C.

$$1404.8\text{ m} + B\cdot\text{Sin}(41.0°) - C\cdot\text{Cos}(35.0°) = 0$$

$$1404.8\text{ m} + (5548.86\text{ m})\cdot\text{Sin}(41.0°) - C\cdot\text{Cos}(35.0°) = 0$$

$$1404.8\text{ m} + 3640.4 = C\cdot\text{Cos}(35.0°)$$

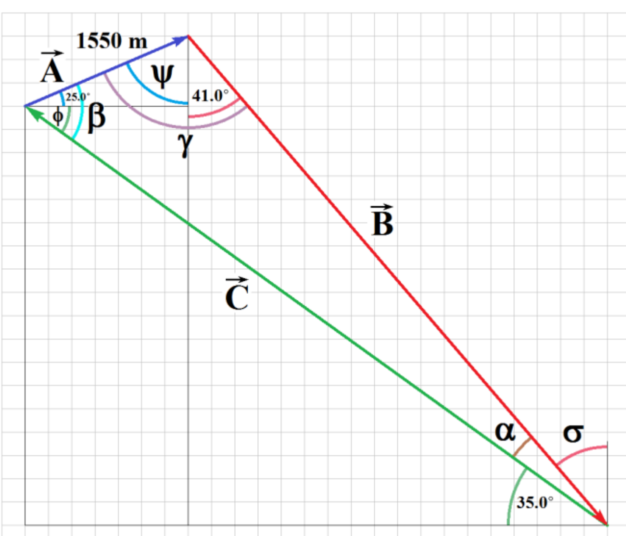
$$5045.2\text{ m} = C\cdot\text{Cos}(35.0°)$$

$$C = 5045.2\text{ m}/\text{Cos}(35.0°) = 6159.1\text{ m} \Rightarrow 6160\text{ m}$$
$$B = 5550\text{ m} \quad \text{and} \quad C = 6160\text{ m}$$

Solution 2 (law of sines)

To use the law of sines we need one length (which we have from side A), and all the interior angles. So, first we need to find α, β, and γ.

Start by adding some extra angles to our diagram.



$\phi = 35.0°$ *{Horizontal lines are parallel}*
$\beta = 25.0° + \phi = 25.0° + 35.0° = 60.0°$

$\psi = 90° - 25.0° = 65.0°$ *{Interior angles}*
$\gamma = \psi + 41.0° = 65.0° + 41.0° = 106.0°$

$\sigma = 41.0°$ *{Vertical lines are parallel}*
$\alpha = 90° - 35.0° - 41.0° = 14.0°$

Check: $60.0° + 106.0° + 14.0° = 180.0°$

$$\frac{\text{Sin }\alpha}{A} = \frac{\text{Sin }\beta}{B} = \frac{\text{Sin }\gamma}{C}$$

$B\cdot\text{Sin}(\alpha) = A\cdot\text{Sin}(\beta)$
$B = A\cdot \text{Sin}(\beta)/\text{Sin}(\alpha) = (1550\text{ m})\cdot\text{Sin}(60.0°)/\text{Sin}(14.0°)$
$B = 5548.65\text{ m} \Rightarrow 5550\text{ m}$

$C\cdot\text{Sin}(\alpha) = A\cdot\text{Sin}(\gamma)$
$C = A\cdot \text{Sin}(\gamma)/\text{Sin}(\alpha) = (1550\text{ m})\cdot\text{Sin}(106.0°)/\text{Sin}(14.0°)$
$C = 6158.83\text{ m} \Rightarrow 6160\text{ m}$

It is usually a good idea to determine what method of solving a problem is the most simple and expedient.

Multiplying Two Vectors

- There are two ways to multiply a vector by another vector.
    - One is called a "cross-product", which produces a vector. ($\vec{A} \times \vec{B}$)
    - One is called a "dot-product", which produces a scalar. ($\vec{A} \cdot \vec{B}$)

    We won't be using cross-products for a while. We will return to this when needed.

- There are two ways to calculate a dot product. One uses magnitudes and angles. The other uses components. Choose the more convenient option.

$$\vec{A} \cdot \vec{B} = AB \cdot \mathrm{Cos}\theta = A_x B_x + A_y B_y + A_z B_z$$

- The angle, θ, is the angle between the two vectors.

$$\mathrm{Cos}\,\theta = \mathrm{Cos}(\theta_A - \theta_B) = \mathrm{Cos}\,(\theta_B - \theta_A)$$

- The square of a vector is considered a dot product with itself. We can represent the magnitude of a vector as a dot product.

$$\vec{A}^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2$$

$$A = |\vec{A}| = \sqrt{\vec{A}^2} = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

- Dot products with unit vectors.
    - The dot product of a unit vector with itself is 1. *(Unit magnitude and $\theta = 0°$)*

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1$$

- The dot product of a unit vector with any other unit vector is 0. *($\theta = 90°$)*

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{\imath} = \hat{\imath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\jmath} = 0$$

- With unit vectors we can multiply products term by term to produce the answer.

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath}) \cdot (B_x\hat{\imath} + B_y\hat{\jmath})$$

$$\vec{A} \cdot \vec{B} = A_x B_x \underbrace{\hat{\imath} \cdot \hat{\imath}}_{1} + A_x B_y \underbrace{\hat{\imath} \cdot \hat{\jmath}}_{0} + A_y B_x \underbrace{\hat{\jmath} \cdot \hat{\imath}}_{0} + A_y B_y \underbrace{\hat{\jmath} \cdot \hat{\jmath}}_{1}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

Example: Find the dot product of $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{B} = 5\hat{\imath} + 12\hat{\jmath}$.

We can do this two ways. In this case, the first (using components) is simplest.

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y = 3 \cdot 5 + 4 \cdot 12 = 15 + 48 = 63$$

The second method requires determining the magnitudes and angles of both vectors.

$$A = |\vec{A}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta_A = \mathrm{Tan}^{-1}\left(\frac{A_y}{A_x}\right) = \mathrm{Tan}^{-1}\left(\frac{4}{3}\right) = 53.13°$$

$$B = |\vec{B}| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

$$\theta_B = \text{Tan}^{-1}\left(\frac{B_y}{B_x}\right) = \text{Tan}^{-1}\left(\frac{12}{5}\right) = 67.38°$$

$$\theta_A - \theta_B = 53.13° - 67.38° = -14.25°$$

$$\vec{A} \cdot \vec{B} = AB \cdot \text{Cos}\theta = 5 \cdot 13 \cdot \text{Cos}(14.25°) = 63$$

Exercises

1. A truck is moving eastward up a hill at 20.0 m/s up a slope of 30.0°. Determine the magnitude of the horizontal and vertical components of its velocity. If we place a coordinate axis with the x-axis horizontal to the east and the y-axis vertically upward, how would we represent the trucks velocity with the unit vectors, $\hat{\imath}$ and $\hat{\jmath}$.

2. Determine the magnitude and angle of vector $\vec{v} = 24.0\,\hat{\imath} - 10.0\,\hat{\jmath}$.

3. Determine the magnitude and angle of vector $\vec{v} = -15.0\,\hat{\imath} + 20.0\,\hat{\jmath}$.

4. Given 3 vectors ($\vec{A} = 13.0 \angle -22.6°$, $\vec{B} = 5.00 \angle 53.1°$, and $\vec{C} = 19.0 \angle 270.0°$), Determine the angle and magnitude of their resultant.

5. Given 3 vectors ($\vec{A} = 13.0 \angle -157.4°$, $\vec{B} = 10.0 \angle 216.9°$, and $\vec{C} = 16.0 \angle 180.0°$), Determine the angle and magnitude of $\vec{D} = 3\vec{A} - \vec{B} + \vec{C}$.

6. Let $\vec{A} = 9.00 \angle 45.0°$ and $\vec{B} = 9.00 \angle -15.0°$. Determine the magnitude of $\vec{A} - \vec{B}$ without using their components.

7. Given 4 vectors ($\vec{A} = 4.00\hat{\imath} + 6.00\,\hat{\jmath}$, $\vec{B} = -2.00\hat{\imath} + 5.00\,\hat{\jmath}$, $\vec{C} = 7.00\hat{\imath} - 2.00\,\hat{\jmath}$, and $\vec{D} = -1.00\hat{\imath} - 3.00\,\hat{\jmath}$), determine the resultant in terms of the unit vectors, $\hat{\imath}$ and $\hat{\jmath}$, and then find the angle and magnitude.

Exercise Solutions

1. A truck is moving eastward up a hill at 20.0 m/s up a slope of 30.0°. Determine the magnitude of the horizontal and vertical components of its velocity. If we place a coordinate axis with the x-axis horizontal to the east and the y-axis vertically upward, how would we represent the trucks velocity with the unit vectors, $\hat{\imath}$ and $\hat{\jmath}$.

   Vertical: $v_y = V \cdot Sin(\theta) = \left(20.0\frac{m}{s}\right) \cdot Sin(30.0°) = 10.0\frac{m}{s}$

   Horizontal: $v_x = V \cdot Cos(\theta) = \left(20.0\frac{m}{s}\right) \cdot Cos(30.0°) = 17.3\frac{m}{s}$

   $\vec{V} = v_x\hat{\imath} + v_y\hat{\jmath} = \left(17.3\frac{m}{s}\right)\hat{\imath} + \left(10.0\frac{m}{s}\right)j$

2. Determine the magnitude and angle of vector $\vec{v} = 24.0\,\hat{\imath} - 10.0\,\hat{\jmath}$.

   $V = |\vec{V}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(24.0)^2 + (10.0)^2} = 26.0$

   I did not include the minus sign on the 10 because it is not needed (it will disappear when the value gets squared). If you put in the minus sign and fail to put parentheses around it, you could erroneously subtract the two terms under the square root instead of adding them.

   $\theta_v = Tan^{-1}\left(\frac{v_y}{v_x}\right) = Tan^{-1}\left(\frac{-10}{24}\right) = -22.6°$

3. Determine the magnitude and angle of vector $\vec{v} = -15.0\,\hat{\imath} + 20.0\,\hat{\jmath}$.

   $V = |\vec{V}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(15.0)^2 + (20.0)^2} = 25.0$

   $\theta_v = 180° + Tan^{-1}\left(\frac{v_y}{v_x}\right) = 180° + Tan^{-1}\left(\frac{20.0}{-15.0}\right) = 126.9°$

   When the x-component is negative, you must add 180° to or subtract 180° from $Tan^{-1}(v_y/v_x)$.

4. Given 3 vectors ($\vec{A} = 13.0\angle -22.6°$, $\vec{B} = 5.00\angle 53.1°$, and $\vec{C} = 19.0\angle 270.0°$), Determine the angle and magnitude of their resultant.

   $A_x = A \cdot Cos(\theta_A) = (13.0) \cdot Cos(-22.6°) = 12.0$

   $A_y = A \cdot Sin(\theta_A) = (13.0) \cdot Sin(-22.6°) = -5.00$

   $B_x = B \cdot Cos(\theta_B) = (5.00) \cdot Cos(53.1°) = 3.00$

   $B_y = B \cdot Sin(\theta_B) = (5.00) \cdot Sin(53.1°) = 4.00$

   $C_x = C \cdot Cos(\theta_C) = (19.0) \cdot Cos(270°) = 0$

   $C_y = C \cdot Sin(\theta_C) = (19.0) \cdot Sin(270°) = -19.0$

   $R_x = A_x + B_x + C_x = 12.0 + 3.00 + 0 = 15.0$

$$R_y = A_y + B_y + C_y = -5.00 + 4.00 - 19.0 = -20.0$$

$$R = |\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{(15.0)^2 + (20.0)^2} = 25.0$$

$$\theta_R = Tan^{-1}\left(\frac{R_y}{R_x}\right) = Tan^{-1}\left(\frac{-20.0}{15.0}\right) = -53.1°$$

$$\vec{R} = 25.0\angle -53.1° = 15.0\hat{\imath} - 20.0\hat{\jmath}$$

5. Given 3 vectors ($\vec{A} = 13.0\angle -157.4°$, $\vec{B} = 10.0\angle 216.9°$, and $\vec{C} = 16.0\angle 180.0°$), Determine the angle and magnitude of $\vec{D} = 3\vec{A} - \vec{B} + \vec{C}$.

$$A_x = A \cdot Cos(\theta_A) = (13.0) \cdot Cos(-157.4°) = -12.0$$

$$A_y = A \cdot Sin(\theta_A) = (13.0) \cdot Sin(-157.4°) = -5.00$$

$$B_x = B \cdot Cos(\theta_B) = (10.0) \cdot Cos(216.9°) = -8.00$$

$$B_y = B \cdot Sin(\theta_B) = (10.0) \cdot Sin(216.9°) = -6.00$$

$$C_x = C \cdot Cos(\theta_C) = (16.0) \cdot Cos(180.0°) = -16.0$$

$$C_y = C \cdot Sin(\theta_C) = (16.0) \cdot Sin(180.0°) = 0$$

$$D_x = 3A_x - B_x + C_x = 3(-12.0) + 8.00 - 16.0 = -44.0$$

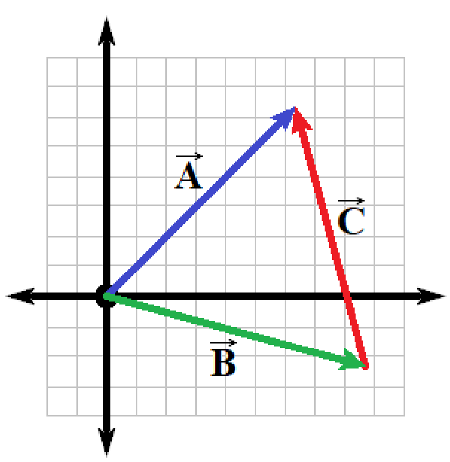
$$D_y = 3A_y - B_y + C_y = 3(-5.00) + 6.00 + 0 = -9.00$$

$$D = |\vec{D}| = \sqrt{D_x^2 + D_y^2} = \sqrt{(44.0)^2 + (9.00)^2} = 44.9$$

$$\theta_D = Tan^{-1}\left(\frac{D_y}{D_x}\right) - 180° = Tan^{-1}\left(\frac{-9.00}{-44.0}\right) - 180° = -168.4°$$

$$\vec{D} = 44.9\angle -168.4° = -44.0\hat{\imath} - 9.00\hat{\jmath}$$

6. Let $\vec{A} = 9.00\angle 45.0°$ and $\vec{B} = 9.00\angle -15.0°$. Determine the magnitude of $\vec{A} - \vec{B}$ without using their components.



Make a diagram with vectors $\vec{A}$ and $\vec{B}$ and then connect the tips with a 3rd vector, $\vec{C}$. It's not immediately evident what $\vec{C}$ represents, but this can be found pretty easily. In the diagram, you can see that $\vec{B}$ and $\vec{C}$ are tip to tail, and that is a sum. The resultant of that sum is $\vec{A}$. This gives you the equation, $\vec{B} + \vec{C} = \vec{A}$. If you subtract $\vec{B}$ from both sides of that equation, you find $\vec{C} = \vec{A} - \vec{B}$. So, $\vec{C}$ is what we are looking for.

Then just use the Law of Cosines to get your answer. The angle between $\vec{A}$ and $\vec{B}$ is just $\theta_A - \theta_B$.

$$C = \sqrt{A^2 + B^2 - 2AB \cdot Cos(\theta_A - \theta_B)}$$

$$C = \sqrt{(9.00)^2 + (9.00)^2 - 2(9.00)(9.00) \cdot Cos[45.0° - (-15.0°)]} = 9.00$$

Alternatively, you could have noted that as the angle between the two sides A and B is 60°, and as A and B have equal lengths the angles opposite them have to be the same. That makes this an equilateral triangle. As all the sides are the same, C must be 9.00.

7. Given 4 vectors ($\vec{A} = 4.00\hat{\imath} + 6.00\,\hat{\jmath}$, $\vec{B} = -2.00\hat{\imath} + 5.00\,\hat{\jmath}$, $\vec{C} = 7.00\hat{\imath} - 2.00\,\hat{\jmath}$, and $\vec{D} = -1.00\hat{\imath} - 3.00\,\hat{\jmath}$), determine the resultant in terms of the unit vectors, $\hat{\imath}$ and $\hat{\jmath}$, and then find the angle and magnitude.

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} = \left(A_x\hat{\imath} + A_y\hat{\jmath}\right) + \left(B_x\hat{\imath} + B_y\hat{\jmath}\right) + \left(C_x\hat{\imath} + C_y\hat{\jmath}\right) + \left(D_x\hat{\imath} + D_y\hat{\jmath}\right)$$

Now just group the x and y-components together, and factor out the unit vectors.

$$\vec{R} = (A_x\hat{\imath} + B_x\hat{\imath} + C_x\hat{\imath} + D_x\hat{\imath}) + \left(A_y\hat{\jmath} + B_y\hat{\jmath} + C_y\hat{\jmath} + D_y\hat{\jmath}\right)$$

$$\vec{R} = (A_x + B_x + C_x + D_x)\hat{\imath} + \left(A_y + B_y + C_y + D_y\right)\hat{\jmath}$$

$$\vec{R} = (4.00 - 2.00 + 7.00 - 1.00)\hat{\imath} + (6.00 + 5.00 - 2.00 - 3.00)\hat{\jmath}$$

$$\vec{R} = 8.00\hat{\imath} + 6.00\hat{\jmath}$$

$$R = \left|\vec{R}\right| = \sqrt{R_x^2 + R_y^2} = \sqrt{(8.00)^2 + (6.00)^2} = 10.0$$

$$\theta_R = Tan^{-1}\left(\frac{R_y}{R_x}\right) = Tan^{-1}\left(\frac{6.00}{8.00}\right) = 36.9°$$

$$\vec{R} = 10.0\angle 36.9° = 8.00\hat{\imath} + 6.00\hat{\jmath}$$